# Interval-Division Numeral Systems

JM

## Abstract

An axiomatic description of what I call interval-division numeral systems. There are five axioms and three theorems. Axiom 1 defines a numeral system as a set of non-negative integer digit sequences, (called numerals), a function mapping each numeral to a sequence of positive integer bases, and a value function that converts numerals and base sequences into (possibly extended) real numbers. Axiom 2 requires that each digit in a numeral must be less than the corresponding base. Axiom 3 requires that numerals that match the first n digits must also match the first n+1 bases. Axiom 4 requires that the set of numerals be non-empty, and has two additional conditions that, in short, allow any reasonably constructed numeral to exist. Axiom 5 requires that every base sequence has at least infinitely many bases greater than 1. Theorem 1 shows that the value of every numeral lies on the closed unit interval [0,1]. Theorem 2 shows that at most two numerals can have the same value. Theorem 3 shows that every number on the closed unit interval [0,1] is the value of some numeral.

For the sake of brevity, remarks and corollaries are not fully justified, but lemmas and theorems are justified.

It is worth noting that the numeral systems described in this paper are not necessarily positional numeral systems, although some are.

(If you notice any errors, please let me know. Thanks.)

## Definition

Let $\mathbb{N}_0$ denote the set of all non-negative integers. Let $\mathbb{N}$ denote the set of all positive integers. Let $\mathbb{N}_0^\infty$ denote the set of all non-negative integer sequences. Let $\mathbb{N}^\infty$ denote the set of all positive integer sequences.

## Axiom 1

Let $\mathcal{N} = (\mathcal{A}, \beta, \phi)$, where $\mathcal{A} \subseteq \mathbb{N}_0^\infty$ is called the numeral space of $\mathcal{N}$, $\beta : \mathcal{A} \to \mathbb{N}^\infty$ is called the base function of $\mathcal{N}$, and $\phi$ is called the valuation function of $\mathcal{N}$, where

$$\phi(\{a_n\}) = \sum_{n=1}^{\infty} \frac{a_n}{\prod_{m=1}^{n} b_m} \tag{1}$$

for every $\{a_n\} \in \mathcal{A}$, where $\{b_n\} = \beta(\{a_n\})$.

## Axiom 2

Let $\{a_n\} \in \mathcal{A}$. Then for every $n \in \mathbb{N}$, $a_n < b_n$, where $\{b_n\} = \beta(\{a_n\})$.

## Axiom 3

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$. Suppose for some $k \in \mathbb{N}$ that $a_n^{(1)} = a_n^{(2)}$ for every $n < k$. Then $b_n^{(1)} = b_n^{(2)}$ for every $n < k + 1$, where $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$ and $\{b_n^{(2)}\} = \beta(\{a_n^{(2)}\})$

## Axiom 4

a) $\mathcal{A}$ is non-empty.
b) Let $\{a_n^{(1)}\} \in \mathcal{A}$. Let $k \in \mathbb{N}$. Let $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$. Then for every $a < b_k^{(1)}$, there exists an $\{a_n^{(2)}\} \in \mathcal{A}$ such that $a_n^{(1)} = a_n^{(2)}$ for every $n < k$ and $a_k^{(2)} = a$.
c) Let $\{a_n\} \in \mathbb{N}_0^\infty$. Suppose for every $k \in \mathbb{N}$ there exists an $\{a_n^{(k)}\} \in \mathcal{A}$ such that $a_n = a_n^{(k)}$ for every $n < k$. Then $\{a_n\} \in \mathcal{A}$.

## Axiom 5

Let $\{a_n\} \in \mathcal{A}$. Let $\{b_n\} = \beta(\{a_n\})$. Then for every $k \in \mathbb{N}$, there exists an $n > k$ such that $b_n > 1$.

## Definition

Call any $\mathcal{N}$ that satisfies Axioms 1-5 an interval-division numeral system.

## Goal

We wish to show that $\phi$ is a function from the numeral space $\mathcal{A}$ into the closed unit interval $[0, 1]$.

## Definition

Define partial valuation functions $\phi_k$ for every $k \in \mathbb{N}_0$, where

$$\phi_k(\{a_n\}) = \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} \tag{2}$$

for every $\{a_n\} \in \mathcal{A}$, where $\{b_n\} = \beta(\{a_n\})$.

## Remark 1.1

$$\phi(\{a_n\}) = \lim_{k \to \infty} \phi_k(\{a_n\}) \tag{3}$$

for every $\{a_n\} \in \mathcal{A}$.

## Remark 1.2

$$\phi_{k+1}(\{a_n\}) \geq \phi_k(\{a_n\}) \tag{4}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

## Lemma 1.3

$$\phi_{k_2}(\{a_n\}) < \phi_{k_1}(\{a_n\}) + \frac{1}{\prod_{m=1}^{k_1} b_m} \tag{5}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k_1, k_2 \in \mathbb{N}_0$ such that $k_2 > k_1$, where $\{b_n\} = \beta(\{a_n\})$.

### Proof

Let $\{a_n\} \in \mathcal{A}$, let $k_1, k_2 \in \mathbb{N}_0$ where $k_2 > k_1$, and let $\{b_n\} = \beta(\{a_n\})$. Then

$$\begin{aligned}
\phi_{k_2}(\{a_n\}) &= \phi_{k_1}(\{a_n\}) + \sum_{n=k_1+1}^{k_2} \frac{a_n}{\prod_{m=1}^{n} b_m} \\
&\leq \phi_{k_1}(\{a_n\}) + \sum_{n=k_1+1}^{k_2} \frac{b_n - 1}{\prod_{m=1}^{n} b_m} && \text{(by Axiom 2)} \\
&= \phi_{k_1}(\{a_n\}) + \sum_{n=k_1+1}^{k_2} \frac{1}{\prod_{m=1}^{n-1} b_m} - \frac{1}{\prod_{m=1}^{n} b_m} \\
&= \phi_{k_1}(\{a_n\}) + \frac{1}{\prod_{m=1}^{k_1} b_m} - \frac{1}{\prod_{m=1}^{k_2} b_m} \\
&< \phi_{k_1}(\{a_n\}) + \frac{1}{\prod_{m=1}^{k_1} b_m}.
\end{aligned}$$

## Corollary 1.4

It follows from Lemma 1.3 that

$$\phi_k(\{a_n\}) < 1 \tag{6}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

## Remark 1.5

$$\phi_k(\{a_n\}) \geq 0 \tag{7}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

## Theorem 1

$\phi(\{a_n\}) \in [0, 1]$ for every $\{a_n\} \in \mathcal{A}$.

### Proof

Let $\{a_n\} \in \mathcal{A}$.

By Remark 1.1, $\phi(\{a_n\}) = \lim_{k\to\infty} \phi_k(\{a_n\})$.

By Remark 1.2, $\phi_{k+1}(\{a_n\}) \geq \phi_k(\{a_n\})$ for every $k \in \mathbb{N}_0$.

By Corollary 1.4, $\phi_k(\{a_n\}) < 1$ for every $k \in \mathbb{N}_0$.

So by Monotone Convergence Theorem, $\phi(\{a_n\})$ exists and $\phi(\{a_n\}) \leq 1$.

By Remark 1.5, $\phi_k(\{a_n\}) \geq 0$ for every $k \in \mathbb{N}_0$.

So $\phi(\{a_n\}) \geq 0$.

Therefore, $\phi(\{a_n\}) \in [0, 1]$.

## Goal

We wish to show that at most two distinct elements $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ can share the same value.

## Definition

Let $\{a_n\} \in \mathcal{A}$. Then say $\{a_n\}$ is terminating if and only if there exists some $k \in \mathbb{N}$ such that $a_n = 0$ for every $n > k$. Otherwise, say $\{a_n\}$ is non-terminating.

## Remark 2.1

$\{a_n\}$ is terminating if and only if there exists some $k \in \mathbb{N}$ such that $\phi(\{a_n\}) = \phi_k(\{a_n\})$.

## Corollary 2.2

It follows from Lemma 1.3 that

$$\phi(\{a_n\}) \leq \phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m} \tag{8}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$, where $\{b_n\} = \beta(\{a_n\})$.

## Lemma 2.3

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$. If there exists some $k \in \mathbb{N}$ such that $a_n^{(1)} = a_n^{(2)}$ for every $n \leq k$, then $\phi_k(\{a_n^{(1)}\}) = \phi_k(\{a_n^{(2)}\})$.

### Proof

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$, let $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$, and let $\{b_n^{(2)}\} = \beta(\{a_n^{(2)}\})$.

Suppose there exists some $k \in \mathbb{N}$ such that $a_n^{(1)} = a_n^{(2)}$ for every $n \leq k$.

Then by Axiom 3, $b_n^{(1)} = b_n^{(2)}$ for every $n \leq k$, So

$$\phi_k(\{a_n^{(1)}\}) = \sum_{n=1}^{k} \frac{a_n^{(1)}}{\prod_{m=1}^{n} b_m^{(1)}} = \sum_{n=1}^{k} \frac{a_n^{(2)}}{\prod_{m=1}^{n} b_m^{(2)}} = \phi_k(\{a_n^{(2)}\}).$$

## Lemma 2.4

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ be two distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$. Then there exists some $k \in \mathbb{N}$ such that either

$$\phi(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(1)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(1)}} = \phi(\{a_n^{(1)}\})$$

or

$$\phi(\{a_n^{(1)}\}) = \phi_k(\{a_n^{(1)}\}) = \phi_k(\{a_n^{(2)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(2)}} = \phi(\{a_n^{(2)}\}).$$

### Proof

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ be distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$, let $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$, and let $\{b_n^{(2)}\} = \beta(\{a_n^{(2)}\})$.

Since $\{a_n^{(1)}\} \neq \{a_n^{(2)}\}$, there exists a least $k \in \mathbb{N}$ such that $a_k^{(1)} \neq a_k^{(2)}$.

So $a_n^{(1)} = a_n^{(2)}$ for every $n < k$.

Therefore, by Lemma 2.3, $\phi_{k-1}(\{a_n^{(1)}\}) = \phi_{k-1}(\{a_n^{(2)}\})$.

And by Axiom 3, $b_n^{(1)} = b_n^{(2)}$ for every $n < k + 1$

Assume $a_k^{(1)} < a_k^{(2)}$. Then

$$\begin{aligned}
\phi_k(\{a_n^{(2)}\}) &= \phi_{k-1}(\{a_n^{(2)}\}) + \frac{a_k^{(2)}}{\prod_{m=1}^{k} b_m^{(2)}} \\
&\geq \phi_{k-1}(\{a_n^{(1)}\}) + \frac{a_k^{(1)} + 1}{\prod_{m=1}^{k} b_m^{(1)}} \\
&= \phi_k(\{a_n^{(1)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(1)}} \\
&\geq \phi(\{a_n^{(1)}\}) && \text{(Corollary 2.2)} \\
&= \phi(\{a_n^{(2)}\}) \\
&\geq \phi_k(\{a_n^{(2)}\}).
\end{aligned}$$

Therefore,

$$\phi(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(1)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(1)}} = \phi(\{a_n^{(1)}\}).$$

Assume instead that $a_k^{(2)} < a_k^{(1)}$. Then by similar logic,

$$\phi(\{a_n^{(1)}\}) = \phi_k(\{a_n^{(1)}\}) = \phi_k(\{a_n^{(2)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(2)}} = \phi(\{a_n^{(2)}\}).$$

## Lemma 2.5

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ be two distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$. Then one numeral is terminating while the other is non-terminating.

## Proof

Let $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ be two distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$, let $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$ and let $\{b_n^{(2)}\} = \beta(\{a_n^{(2)}\})$.

Without loss of generality, by Lemma 2.4, assume that

$$\phi(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(1)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(1)}} = \phi(\{a_n^{(1)}\})$$

for some $k \in \mathbb{N}$.

Then $\phi(\{a_n^{(2)}\}) = \phi_k(\{a_n^{(2)}\})$.

So by Remark 2.1, $\{a_n^{(2)}\}$ is terminating.

By Remark 1.2 and Lemma 1.3, $\phi_{k'}(\{a_n^{(1)}\}) < \phi_k(\{a_n^{(1)}\}) + \frac{1}{\prod_{m=1}^{k} b_m^{(1)}}$ for all $k' \in \mathbb{N}$.

So $\phi_{k'}(\{a_n^{(1)}\}) < \phi(\{a_n^{(1)}\})$ for all $k' \in \mathbb{N}$.

So by Remark 2.1, $\{a_n^{(1)}\}$ is non-terminating.

# Theorem 2

At most two distinct numerals $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ can share the same value $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$.

## Proof

Suppose there exist three distinct numerals $\{a_n^{(1)}\}, \{a_n^{(2)}\}, \{a_n^{(3)}\} \in \mathcal{A}$ such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\}) = \phi(\{a_n^{(3)}\})$.

Then $\{a_n^{(1)}\}, \{a_n^{(2)}\} \in \mathcal{A}$ are two distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\})$.

Without loss of generality, by Lemma 2.5, assume $\{a_n^{(1)}\}$ is non-terminating, and $\{a_n^{(2)}\}$ is terminating.

$\{a_n^{(1)}\}, \{a_n^{(3)}\} \in \mathcal{A}$ are two distinct numerals such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(3)}\})$.

So by Lemma 2.5, since $\{a_n^{(1)}\}$ is non-terminating, $\{a_n^{(3)}\}$ is terminating.

$\{a_n^{(2)}\}, \{a_n^{(3)}\} \in \mathcal{A}$ are two distinct numerals such that $\phi(\{a_n^{(2)}\}) = \phi(\{a_n^{(3)}\})$.

So by Lemma 2.5, since $\{a_n^{(2)}\}$ is terminating, $\{a_n^{(3)}\}$ is non-terminating.

Therefore $\{a_n^{(3)}\}$ is both terminating and non-terminating. (Contradiction.)

Therefore, there do not exist three distinct numerals $\{a_n^{(1)}\}, \{a_n^{(2)}\}, \{a_n^{(3)}\} \in \mathcal{A}$ such that $\phi(\{a_n^{(1)}\}) = \phi(\{a_n^{(2)}\}) = \phi(\{a_n^{(3)}\})$.

# Goal

We wish to show that $\phi$ is onto.

## Lemma 3.1

There exists a function $\alpha_T : [0,1) \to \mathcal{A}$ such that $\phi(\alpha_T(x)) = x$ for every $x \in [0,1)$.

## Proof

Let $x \in [0,1)$.

Define the sequences $\{a_n\}$, $\{b_n\}$ and $\{x_n\}$ as follows:

### Case k = 1

By Axiom 4.a, there exists some $\{a_n^{(0)}\} \in \mathcal{A}$.

Let $\{b_n^{(0)}\} = \beta(\{a_n^{(0)}\})$.

Let $b_1 = b_1^{(0)}$.

Let $x_1 = x$.

Let $a_1 = \max\{a \in \mathbb{N}_0 | \frac{a}{b_1} \leq x_1\}$.

$x_1 \geq 0$, so $\{a \in \mathbb{N}_0 | \frac{a}{b_1} \leq x_1\}$ is non-empty, and $a_1$ is well-defined.

$x_1 < 1$, so $a_1 < b_1$.

So by Axioms 3 and 4.b, there exists some $\{a_n^{(1)}\} \in \mathcal{A}$ such that $a_1^{(1)} = a_1$ and $b_1^{(1)} = b_1$, where $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$.

### Inductive Step

Let $k \in \mathbb{N}$.

Assume that $a_n$, $b_n$, and $x_n$ have been chosen for all $n \leq k$.

Assume that $x_n \in [0,1)$ for all $n \leq k$.

Assume that we have found some $\{a_n^{(k)}\} \in \mathcal{A}$ and $\{b_n^{(k)}\} = \beta(\{a_n^{(k)}\})$ such that $a_n^{(k)} = a_n$ and $b_n^{(k)} = b_n$ for all $n \leq k$.

Let $b_{k+1} = b_{k+1}^{(k)}$.

Let $x_{k+1} = b_k x_k - a_k$.

Let $a_{k+1} = \max\{a \in \mathbb{N}_0 | \frac{a}{b_{k+1}} \leq x_{k+1}\}$.

It is trivial to show that $x_{k+1} \in [0,1)$.

$x_{k+1} \geq 0$, so $\{a \in \mathbb{N}_0 | \frac{a}{b_{k+1}} \leq x_{k+1}\}$ is non-empty, and $a_{k+1}$ is well-defined.

$x_{k+1} < 1$, so $a_{k+1} < b_{k+1}$.

So by Axioms 3 and 4.b, there exists some $\{a_n^{(k+1)}\} \in \mathcal{A}$ such that $a_n^{(k+1)} = a_n$ and $b_n^{(k+1)} = b_n$ for every $n \leq k+1$, where $\{b_n^{(k+1)}\} = \beta(\{a_n^{(k+1)}\})$.

### Induction

Therefore, by induction, for every $k \in \mathbb{N}$, there exist $\{a_n^{(k)}\} \in \mathcal{A}$ and $\{b_n^{(k)}\} = \beta(\{a_n^{(k)}\})$ such that $a_n^{(k)} = a_n$ and $b_n^{(k)} = b_n$ for all $n \leq k$.

So by Axioms 3 and 4.c, $\{a_n\} \in \mathcal{A}$, and $\beta(\{a_n\}) = \{b_n\}$.

Define $\alpha_T(x) = \{a_n\}$.

It is left as an exercise to show that $\phi(\alpha_T(x)) = x$. (Requires Axiom 5)

## Lemma 3.2

There exists a function $\alpha_N : (0,1] \to \mathcal{A}$ such that $\phi(\alpha_N(x)) = x$ for every $x \in (0,1]$.

## Proof

Let $x \in (0,1]$.

Define the sequences $\{a_n\}$, $\{b_n\}$ and $\{x_n\}$ as follows:

### Case k = 1

By Axiom 4.a, there exists some $\{a_n^{(0)}\} \in \mathcal{A}$.

Let $\{b_n^{(0)}\} = \beta(\{a_n^{(0)}\})$.

Let $b_1 = b_1^{(0)}$.

Let $x_1 = x$.

Let $a_1 = \max\{a \in \mathbb{N}_0 | \frac{a}{b_1} < x_1\}$.

$x_1 > 0$, so $\{a \in \mathbb{N}_0 | \frac{a}{b_1} < x_1\}$ is non-empty, and $a_1$ is well-defined.

$x_1 \leq 1$, so $a_1 < b_1$.

So by Axioms 3 and 4.b, there exists some $\{a_n^{(1)}\} \in \mathcal{A}$ such that $a_1^{(1)} = a_1$ and $b_1^{(1)} = b_1$, where $\{b_n^{(1)}\} = \beta(\{a_n^{(1)}\})$.

### Inductive Step

Let $k \in \mathbb{N}$.

Assume that $a_n$, $b_n$, and $x_n$ have been chosen for all $n \leq k$.

Assume that $x_n \in (0,1]$ for all $n \leq k$.

Assume that we have found some $\{a_n^{(k)}\} \in \mathcal{A}$ and $\{b_n^{(k)}\} = \beta(\{a_n^{(k)}\})$ such that $a_n^{(k)} = a_n$ and $b_n^{(k)} = b_n$ for all $n \leq k$.

Let $b_{k+1} = b_{k+1}^{(k)}$.

Let $x_{k+1} = b_k x_k - a_k$.

Let $a_{k+1} = \max\{a \in \mathbb{N}_0 | \frac{a}{b_{k+1}} < x_{k+1}\}$.

It is trivial to show that $x_{k+1} \in (0,1]$.

$x_{k+1} > 0$, so $\{a \in \mathbb{N}_0 | \frac{a}{b_{k+1}} < x_{k+1}\}$ is non-empty, and $a_{k+1}$ is well-defined.

$x_{k+1} \leq 1$, so $a_{k+1} < b_{k+1}$.

So by Axioms 3 and 4.b, there exists some $\{a_n^{(k+1)}\} \in \mathcal{A}$ such that $a_n^{(k+1)} = a_n$ and $b_n^{(k+1)} = b_n$ for every $n \leq k+1$, where $\{b_n^{(k+1)}\} = \beta(\{a_n^{(k+1)}\})$.

### Induction

Therefore, by induction, for every $k \in \mathbb{N}$, there exist $\{a_n^{(k)}\} \in \mathcal{A}$ and $\{b_n^{(k)}\} = \beta(\{a_n^{(k)}\})$ such that $a_n^{(k)} = a_n$ and $b_n^{(k)} = b_n$ for all $n \leq k$.

So by Axioms 3 and 4.c, $\{a_n\} \in \mathcal{A}$, and $\beta(\{a_n\}) = \{b_n\}$.

Define $\alpha_N(x) = \{a_n\}$.

It is left as an exercise to show that $\phi(\alpha_N(x)) = x$. (Requires Axiom 5)

## Definition

The function $\alpha_T$ from the proof of Lemma 3.1 is called the terminable division algorithm. The function $\alpha_N$ from the proof of Lemma 3.2 is called the non-terminable division algorithm.

## Note

The key difference between $\alpha_T$ and $\alpha_N$ is that $a_k = \max\{a \in \mathbb{N}_0 | \frac{a}{b_k} \leq x_k\}$ for $\alpha_T$, but $a_k = \max\{a \in \mathbb{N}_0 | \frac{a}{b_k} < x_k\}$ for $\alpha_N$.

## Theorem 3

For every $x \in [0, 1]$, there exists some $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.

### Proof

Let $x \in [0, 1]$.
Then $x \in [0, 1) \cup (0, 1]$.
Suppose $x \in [0, 1)$.
Then by Lemma 3.1, there exists some $\{a_n\} = \alpha_T(x)$ such that $\phi(\{a_n\}) = x$.
Suppose $x \in (0, 1]$.
Then by Lemma 3.2, there exists some $\{a_n\} = \alpha_N(x)$ such that $\phi(\{a_n\}) = x$.
Therefore, for all $x \in [0, 1]$, there exists some $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.

## Remark 3.3

$\alpha_N(x)$ is always non-terminating. $\alpha_T(x)$ is terminating if and only if there is some terminating $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.

## Corollary 3.4

As a consequence of Theorem 2 and Remark 3.3, $\{a_n\} \in \mathcal{A}$ if and only if $\{a_n\} = \alpha_T(x)$ or $\{a_n\} = \alpha_N(x)$ for some $x \in [0, 1]$.