# Interval-Division Numeral Systems (Supplement)

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## Abstract

A companion to Interval-Division Numeral Systems. A working out of all remarks, corollaries, and exercises for the reader.

(If you notice any errors, please let me know. Thanks.)

## Remark 1.1

$$\phi(\{a_n\}) = \lim_{k\to\infty} \phi_k(\{a_n\}) \tag{1}$$

for every $\{a_n\} \in \mathcal{A}$.

### Proof

$$\phi(\{a_n\}) = \sum_{n=1}^{\infty} \frac{a_n}{\prod_{m=1}^{n} b_m} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} = \lim_{k\to\infty} \phi_k(\{a_n\})$$

## Remark 1.2

$$\phi_{k+1}(\{a_n\}) \geq \phi_k(\{a_n\}) \tag{2}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

### Proof

$$\begin{aligned}
\phi_{k+1}(\{a_n\}) &= \sum_{n=1}^{k+1} \frac{a_n}{\prod_{m=1}^{n} b_m} \\
&= \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} + \frac{a_k}{\prod_{m=1}^{k} b_m} \\
&= \phi_k(\{a_n\}) + \frac{a_k}{\prod_{m=1}^{k} b_m}.
\end{aligned}$$

$$\begin{aligned}
\frac{a_k}{\prod_{m=1}^{k} b_m} &\geq 0, \\
\phi_k(\{a_n\}) + \frac{a_k}{\prod_{m=1}^{k} b_m} &\geq \phi_k(\{a_n\}), \\
\phi_{k+1}(\{a_n\}) &\geq \phi_k(\{a_n\}).
\end{aligned}$$

## Corollary 1.4

It follows from Lemma 1.3 that

$$\phi_k(\{a_n\}) < 1 \tag{3}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

## Proof

Let $\{a_n\} \in \mathcal{A}$ and $k \in \mathbb{N}_0$

Suppose $k = 0$.

Then $\phi_k(\{a_n\}) = 0 < 1$.

Suppose $k > 0$.

Then by Lemma 1.3, $\phi_k(\{a_n\}) < \phi_0(\{a_n\}) + \frac{1}{\prod_{m=1}^{0} b_m} = 0 + 1 = 1$.

So $\phi_k(\{a_n\}) < 1$ for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

# Remark 1.5

$$\phi_k(\{a_n\}) \geq 0 \tag{4}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$.

## Proof

By definition, $a_i \geq 0$ and $b_i > 0$ for all $i \in \mathbb{N}$.

So $\prod_{m=1}^{n} b_m > 0$ for all $n \in \mathbb{N}$.

So $\frac{a_n}{\prod_{m=1}^{n} b_m} \geq 0$ for all $n \in \mathbb{N}$.

So $\sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} \geq 0$ for all $k \in \mathbb{N}$.

Or $\phi_k(\{a_n\}) \geq 0$ for all $k \in \mathbb{N}$.

Suppose $k = 0$.

Then $\phi_k(\{a_n\}) = 0$.

So $\phi_k(\{a_n\}) \geq 0$ for all $k \in \mathbb{N}_0$.

# Remark 2.1

$\{a_n\}$ is terminating if and only if there exists some $k \in \mathbb{N}$ such that $\phi(\{a_n\}) = \phi_k(\{a_n\})$.

## Proof

Let $\{a_n\} \in \mathcal{A}$, $\{b_n\} = \beta(\{a_n\})$.

Suppose $\{a_n\}$ is terminating.

Then there exists some $k \in \mathbb{N}$ such that $a_n = 0$ for every $n > k$. So

$$\begin{aligned}\phi(\{a_n\}) &= \sum_{n=1}^{\infty} \frac{a_n}{\prod_{m=1}^{n} b_m} \\ &= \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} + \sum_{n=k+1}^{\infty} \frac{0}{\prod_{m=1}^{n} b_m} \\ &= \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} \\ &= \phi_k(\{a_n\}).\end{aligned}$$

Suppose $\{a_n\}$ is non-terminating.
Let $k \in \mathbb{N}$.
Since $\{a_n\}$ is non-terminating, there exists some $k' > k$ such that $a_{k'} \neq 0$.
So $\frac{a_{k'}}{\prod_{m=1}^{k'} b_m} > 0$. Therefore,

$$\begin{aligned}\phi_k(\{a_n\}) &= \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} \\ &< \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m} + \frac{a_{k'}}{\prod_{m=1}^{k'} b_m} \\ &\leq \sum_{n=1}^{k'} \frac{a_n}{\prod_{m=1}^{n} b_m} \\ &\leq \sum_{n=1}^{\infty} \frac{a_n}{\prod_{m=1}^{n} b_m} \\ &= \phi(\{a_n\}).\end{aligned}$$

So $\phi_k(\{a_n\}) \neq \phi(\{a_n\})$.

## Corollary 2.2

It follows from Lemma 1.3 that

$$\phi(\{a_n\}) \leq \phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m} \tag{5}$$

for every $\{a_n\} \in \mathcal{A}$ and every $k \in \mathbb{N}_0$, where $\{b_n\} = \beta(\{a_n\})$.

### Proof

Let $k \in \mathbb{N}_0$.
By Lemma 1.3, $\phi_{k'}(\{a_n\}) < \phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m}$ for every $k' > k$.

So it follows from Remark 1.2 that $\phi_{k'}(\{a_n\}) < \phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m}$ for every $k' \in \mathbb{N}_0$.

So $\phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m}$ is an upper bound for the set of all $\phi_{k'}(\{a_n\})$ such that $k' \in \mathbb{N}_0$.

By Monotone Convergence Theorem, $\phi(\{a_n\})$ is the least upper bound of the set of all $\phi_{k'}(\{a_n\})$ such that $k' \in \mathbb{N}_0$.

So $\phi(\{a_n\}) \leq \phi_k(\{a_n\}) + \frac{1}{\prod_{m=1}^{k} b_m}$.

# Lemma 3.1.a

It is trivial to show that $x_{k+1} \in [0, 1)$

## Proof

Suppose $x_{k+1} < 0$.
Then $b_k x_k - a_k < 0$
So $x_k < \frac{a_k}{b_k}$.
But by our choice of $a_k$, $x_k \geq \frac{a_k}{b_k}$. This is a contradiction.
So $x_{k+1} \geq 0$.
Suppose $x_{k+1} \geq 1$.
Then $b_k x_k - a_k \geq 1$.
So $x_k \geq \frac{a_k+1}{b_k}$.
But $a_k + 1 > a_k$, and $a_k + 1 \in \mathbb{N}_0$.
So $a_k \neq \max\{a \in \mathbb{N}_0 | \frac{a}{b_k} \leq x_k\}$. This is a contradiction.
Therefore, $x_{k+1} < 1$.
So $x_{k+1} \in [0, 1)$.

# Lemma 3.1.b

$\phi(\alpha_T(x)) = x$

## Proof

### Case k = 1

By definition, $x_1 = x$.
So by definition, $x_2 = b_1 x - a_1$.
Therefore, $\frac{x_2}{b_1} = x - \frac{a_1}{b_1}$.

### Inductive Step

Let $k \in \mathbb{N}$.
Suppose $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
By definition $x_{k+2} = b_{k+1} x_{k+1} - a_{k+1}$.
So $x_{k+1} = \frac{x_{k+2} + a_{k+1}}{b_{k+1}}$.

So $\frac{x_{k+2}+a_{k+1}}{\prod_{m=1}^{k+1} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
Therefore $\frac{x_{k+2}}{\prod_{m=1}^{k+1} b_m} = x - \sum_{n=1}^{k+1} \frac{a_n}{\prod_{m=1}^{n} b_m}$.

### Induction

By induction, for every $k \in \mathbb{N}$, $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
In other words, $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \phi_k(\alpha_T(x))$.
Since $x_{k+1} \in [0, 1)$ for all $k \in \mathbb{N}$,

$$\frac{0}{\prod_{m=1}^{k} b_m} \leq \frac{x_{k+1}}{\prod_{m=1}^{k} b_m} < \frac{1}{\prod_{m=1}^{k} b_m}.$$

By Axiom 5, for every $k \in \mathbb{N}$, there exists an $n > k$ such that $b_n > 1$.
So $\lim_{k\to\infty} \frac{1}{\prod_{m=1}^{k} b_m} = 0$.
So by Squeeze Theorem, $\lim_{k\to\infty} \frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = 0$.
Therefore, $0 = x - \phi(\alpha_T(x))$,
So $\phi(\alpha_T(x)) = x$.

## Lemma 3.2.a

It is trivial to show that $x_{k+1} \in (0, 1]$.

## Proof

Suppose $x_{k+1} \leq 0$.
Then $b_k x_k - a_k \leq 0$
So $x_k \leq \frac{a_k}{b_k}$.
But by our choice of $a_k$, $x_k > \frac{a_k}{b_k}$. This is a contradiction.
So $x_{k+1} > 0$.
Suppose $x_{k+1} > 1$.
Then $b_k x_k - a_k > 1$.
So $x_k > \frac{a_k+1}{b_k}$.
But $a_k + 1 > a_k$, and $a_k + 1 \in \mathbb{N}_0$.
So $a_k \neq \max\{a \in \mathbb{N}_0 | \frac{a}{b_k} < x_k\}$. This is a contradiction.
Therefore, $x_{k+1} \leq 1$.
So $x_{k+1} \in (0, 1]$.

# Lemma 3.2.b

$\phi(\alpha_N(x)) = x$

## Proof

### Case k = 1

By definition, $x_1 = x$.
So by definition, $x_2 = b_1 x - a_1$.
Therefore, $\frac{x_2}{b_1} = x - \frac{a_1}{b_1}$.

### Inductive Step

Let $k \in \mathbb{N}$.
Suppose $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
By definition $x_{k+2} = b_{k+1} x_{k+1} - a_{k+1}$.
So $x_{k+1} = \frac{x_{k+2} + a_{k+1}}{b_{k+1}}$.
So $\frac{x_{k+2} + a_{k+1}}{\prod_{m=1}^{k+1} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
Therefore $\frac{x_{k+2}}{\prod_{m=1}^{k+1} b_m} = x - \sum_{n=1}^{k+1} \frac{a_n}{\prod_{m=1}^{n} b_m}$.

### Induction

By induction, for every $k \in \mathbb{N}$, $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \sum_{n=1}^{k} \frac{a_n}{\prod_{m=1}^{n} b_m}$.
In other words, $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \phi_k(\alpha_N(x))$.
Since $x_{k+1} \in (0, 1]$ for all $k \in \mathbb{N}$,

$$\frac{0}{\prod_{m=1}^{k} b_m} < \frac{x_{k+1}}{\prod_{m=1}^{k} b_m} \leq \frac{1}{\prod_{m=1}^{k} b_m}.$$

By Axiom 5, for every $k \in \mathbb{N}$, there exists an $n > k$ such that $b_n > 1$.
So $\lim_{k \to \infty} \frac{1}{\prod_{m=1}^{k} b_m} = 0$.
So by Squeeze Theorem, $\lim_{k \to \infty} \frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = 0$.
Therefore, $0 = x - \phi(\alpha_N(x))$,
So $\phi(\alpha_N(x)) = x$.

# Remark 3.3.a

$\alpha_N(x)$ is always non-terminating.

## Proof

Let $k \in \mathbb{N}$.
By the proof of Lemma 3.2.b, $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} = x - \phi_k(\alpha_N(x))$.
And $x_{k+1} \in (0, 1]$.
So $x_{k+1} > 0$,
So $\frac{x_{k+1}}{\prod_{m=1}^{k} b_m} > 0$
So $x - \phi_k(\alpha_N(x)) > 0$.

Therefore, $\phi_k(\alpha_N(x)) \neq x = \phi(\alpha_N(x))$.
So by Remark 2.1, $\alpha_N(x)$ is non-terminating.

## Remark 3.3.b

$\alpha_T(x)$ is terminating if and only if there is some terminating $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.

### Proof

Suppose that there is no terminating $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.
If $\alpha_T(x) \in \mathcal{A}$, then $\phi(\alpha_T(x)) = x$.
So $\alpha_T(x)$ cannot be terminating.
Suppose that there exists some terminating $\{a_n\} \in \mathcal{A}$ such that $\phi(\{a_n\}) = x$.
Assume $x = 1$.
Then by Corollary 1.4, $\phi_k(\{a_n\}) < x$ for all $k \in \mathbb{N}_0$.
So $\{a_n\}$ is non-terminating, and, by contradiction, $x \neq 1$.
Therefore, $x \in [0, 1)$.
So by Lemma 3.1, $\alpha_T(x) \in \mathcal{A}$.
Assume $\alpha_T(x) \neq \{a_n\}$
Let $\{b_n\} = \beta(\{a_n\})$.
Then by Lemma 2.4, there exists some $k \in \mathbb{N}$ such that $x - \phi_k(\alpha_T(x)) = \frac{1}{\prod_{m=1}^{k} b_m}$.
But by the proof of Lemma 3.1.b, there exists some $x_{k+1} \in [0, 1)$ such that $x - \phi_k(\alpha_T(x)) = \frac{x_{k+1}}{\prod_{m=1}^{k} b_m}$.
So $x_{k+1} = 1 \notin [0, 1)$. (Contradiction.)
Therefore $\alpha_T(x) = \{a_n\}$.
So $\alpha_T(x)$ is terminating.

## Corollary 3.4

As a consequence of Theorem 2 and Remark 3.3, $\{a_n\} \in \mathcal{A}$ if and only if $\{a_n\} = \alpha_T(x)$ or $\{a_n\} = \alpha_N(x)$ for some $x \in [0, 1]$.

### Proof

Suppose $\{a_n\} = \alpha_T(x)$ or $\{a_n\} = \alpha_N(x)$ for some $x \in [0, 1]$.
Then by Lemmas 3.1 and 3.2, $\{a_n\} \in \mathcal{A}$.

Suppose $\{a_n\} \in \mathcal{A}$.
Then $\phi(\{a_n\}) = x$ for some $x \in [0, 1]$.

Suppose $x = 1$.
Then by Corollary 1.4, $\phi_k(\{a_n\}) < x$ for all $k \in \mathbb{N}_0$.

So by Remark 2.1, $\{a_n\}$ is non-terminating.
By Lemma 3.2, $\alpha_N(x) \in \mathcal{A}$.
By Remark 3.3, $\alpha_N(x)$ is non-terminating.
So by Lemma 2.5, $\{a_n\} = \alpha_N(x)$.

Suppose $x = 0$
Then by Remark 1.5, $\phi_k(\{a_n\}) \geq x$ for all $k \in \mathbb{N}_0$.
And by Monotone Convergence Theorem, $\phi_k(\{a_n\}) \leq x$ for all $k \in \mathbb{N}_0$.
So $\phi_k(\{a_n\}) = x$ for all $k \in \mathbb{N}_0$.
So by Remark 3.2, $\{a_n\}$ is terminating.
By Lemma 3.1, $\alpha_T(x) \in \mathcal{A}$.
By Remark 3.3, since $\{a_n\}$ is terminating, $\alpha_T(x)$ is terminating.
So by Lemma 2.5, $\{a_n\} = \alpha_T(x)$.

Suppose $x \in (0, 1)$.
Then by Lemmas 3.1 and 3.2, $\alpha_T(x), \alpha_N(x) \in \mathcal{A}$.
Suppose $\{a_n\}$ is terminating.
Then by Remark 3.3, since $\{a_n\}$ is terminating, $\alpha_T(x)$ is terminating.
So by Lemma 2.5, $\{a_n\} = \alpha_T(x)$.
Suppose $\{a_n\}$ is non-terminating.
By Remark 3.3, $\alpha_N(x)$ is non-terminating.
So by Lemma 2.5, $\{a_n\} = \alpha_N(x)$.